Application of Geometric Inversion to the Eccentric Annulus System

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The most appropriate coordinate system to be used in analyzing fluid flow and/or heat transfer in an eccentric annulus is that of bipolar coordinates. However, in this coordinate system the solution of the equations of change appears to be lengthy and quite difficult if not impossible to obtain. Thus, problems involving an eccentric annulus are solved approximately using cylindrical coordinates (Cheng and Hwang, 1968; Trombetta, 1971; Yao, 1980; Prusa and Yao, 1983).

The purpose of this note is to present a new approach in extrapolating the experimental data for laminar and turbulent flows in eccentric annuli by the use of an inversion technique. The geometric inversion transforms the eccentric annulus system to the symmetrical case, that is, the concentric system. This transformation is both involutoric and isogonal. In this way, a rather complicated problem in bipolar coordinates can be solved easily in cylindrical coordinates. This method was first used by Nickel (1980) in analyzing the harmonics created by a circular drumhead set in vibration by an impulse at a point, not at the center.

In the following, the conditions under which the inversion technique is applicable in the calculation of the maximum velocity locus are first determined. The technique is then extended to predict the velocity contours in laminar and turbulent flows. The results are shown to be in perfect agreement with the experimental results of Wolffe and Clump (1963) and Jonsson and Sparrow (1966).

The Bipolar Coordinate System

The bipolar coordinates η, ξ, z are related to the rectangular coordinates x, y, z as follows:

$$x = \frac{a \sinh \eta}{\cosh \eta - \cos \xi} \tag{1}$$

$$y = \frac{a \sin \xi}{\cosh \eta - \cos \xi} \tag{2}$$

$$z = z \tag{3}$$

$$\eta = \frac{1}{2} \ln \frac{(x + a)^2 + y^2}{(x - a)^2 + y^2} \tag{4}$$

$$\xi = \arctan \frac{2ay}{x^2 + y^2 - a^2} \tag{5}$$

$$z = z \tag{6}$$

In this coordinate system the lines of constant η represent circles with an equation of the form

$$(x - a \coth \eta)^2 + y^2 = a^2 \operatorname{csch}^2 \eta \tag{7}$$

Thus, the inner and outer surfaces of the annulus are at $\eta = \eta_i$ and $\eta = \eta_o$, respectively, as shown in Figure 1. On the other hand, the lines of constant ξ are given as

$$x^2 + (y - a \cot \xi)^2 = a^2 \csc^2 \xi \tag{8}$$

The eccentric annulus system is identified by two parameters, the radius ratio r_e^*, and the eccentricity ratio ϵ. These are defined by

$$r_e^* = r_{e_i}/r_{e_o} \tag{9}$$

$$\epsilon = \frac{e}{r_{e_o} - r_{e_i}} = \frac{\sinh (\eta_i - \eta_o)}{\sinh \eta_i - \sinh \eta_o} \tag{10}$$

Once the value of r_e^* and ϵ are known, constant η lines representing the inner and outer surfaces of the annulus can be calculated using the following equations:

$$\cosh \eta_i = \frac{(1 + r_e^*) - \epsilon^2(1 - r_e^*)}{2\epsilon r_e^*} \tag{11}$$

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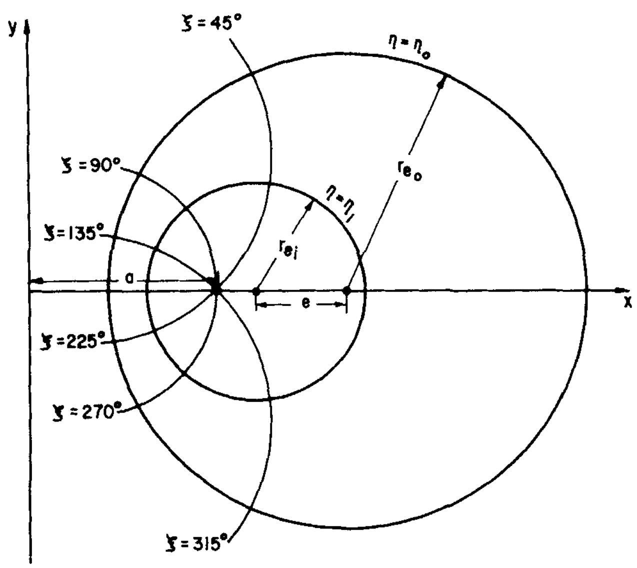

Figure 1. Geometry of eccentric annulus system.

$$\cosh \eta_o = \frac{(1 + r_e^*) + \epsilon^2(1 - r_e^*)}{2\epsilon} \tag{12}$$

Geometric Inversion

The details of geometric inversion are given by Eves (1972). When the point $A(-a, 0)$ is taken as the center of inversion, the Apollonian circles representing constant η lines can be transformed into a set of concentric circles with center at $D(d, 0)$ and radius r_c given by

$$r_c = (a + d)\exp(-\eta) \tag{13}$$

On the other hand, the circles representing constant ξ lines transform into radial lines corresponding to the lines θ = constant in cylindrical coordinates.

The Locus of Maximum Velocity

The inversion technique can predict the locus of maximum velocity in an eccentric annulus if this locus coincides with the constant η line, i.e., $\eta = \eta_{max}$. Therefore, η_{max} values must be independent of ξ. This can be checked by using one of the several analytical expressions available for the laminar velocity distribution in an eccentric annulus (Piercy et al., 1933; Heyda, 1959; Snyder and Goldstein, 1965). The expression given by Snyder and Goldstein is

$$\frac{v_z}{\frac{a^2}{\mu}\left(-\frac{dP}{dz}\right)} = B - C\eta - \frac{\coth \eta}{2} + \sum_{n=1}^{\infty}\left[-F_n e^{n\eta} + (G_n - \coth \eta)\, e^{-n\eta}\right]\cos n\xi \tag{14}$$

where

$$B = \frac{\eta_i \coth \eta_o - \eta_o \coth \eta_i}{2(\eta_i - \eta_o)} \tag{15}$$

$$C = \frac{\coth \eta_o - \coth \eta_i}{2(\eta_i - \eta_o)} \tag{16}$$

$$F_n = \frac{\coth \eta_o - \coth \eta_i}{\exp(2n\eta_i) - \exp(2n\eta_o)} \tag{17}$$

$$G_n = \frac{\coth \eta_o \exp(2n\eta_i) - \coth \eta_i \exp(2n\eta_o)}{\exp(2n\eta_i) - \exp(2n\eta_o)} \tag{18}$$

η_{max} values can be evaluated as a function of ξ by setting $\partial v_z/\partial \eta = 0$. If, however, η_{max} is independent of ξ, then the following relationships should hold:

$$\sinh \eta_{max} = (2C)^{-1/2} \tag{19}$$

$$[2C - n(G_n - \sqrt{1 + 2C})]\, e^{-n\eta_{max}} - nF_n e^{n\eta_{max}} = 0 \quad \text{for } n = 1, 2, 3, \ldots \tag{20}$$

For a given ϵ and r_e^*, the η_{max} value can be calculated from Eq. 19. This value is then substituted into Eq. 20 to check whether it is satisfied. In this way, the region of the applicability of the approximation presented in this work, as the values of the parameters ϵ and r_e^* vary, is determined. It is shown in Figure 2.

In the region where η_{max} is independent of ξ, the equation

$$1 - \exp(-2\eta_{max}) \simeq \{[1 - \exp(-2\eta_i)][1 - \exp(-2\eta_o)]\}^{1/2} \tag{21}$$

can be considered valid. Inversion of Eq. 19 according to Eq. 13 and the use of Eq. 21 gives

$$r_{c_{max}} = r_{c_o}\left[\frac{1 - r_c^{*2}}{2\ln(1/r_c^*)}\right]^{1/2} \tag{22}$$

where

$$r_c^* = r_{c_i}/r_{c_o} \tag{23}$$

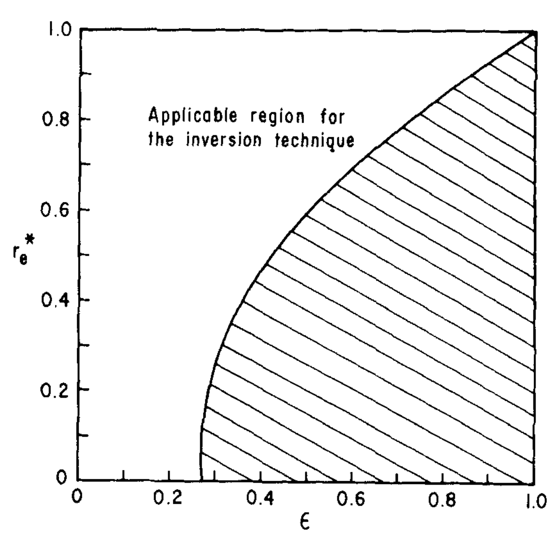

Figure 2. Region of applicability for inversion technique.

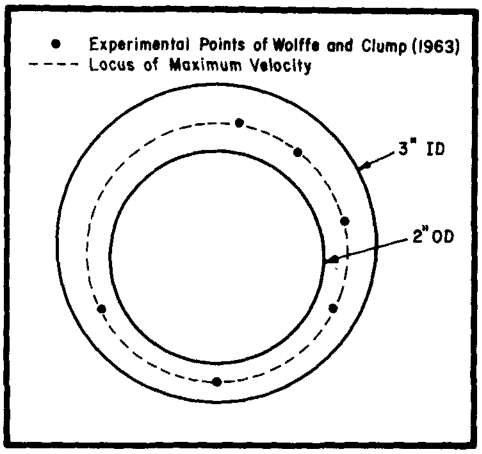

Figure 3. Locus of maximum velocity in an eccentric annulus.

Note that Eq. 22 is equivalent to the expression given by Bird et al. (1960) for laminar flow in a concentric annulus with inner and outer radii of r_{c_i} and r_{c_o}, respectively.

Experimental velocity profile measurements have suggested that the radius of maximum velocity for laminar flow in a concentric annulus is approximately identical to the radius of maximum velocity for turbulent flow (Rothfus et al., 1950, 1955; Walker et al., 1957). The same assumption is also assumed to hold in an eccentric annulus by Heyda (1958). Therefore, Eq. 22 will be considered valid for both laminar and turbulent flows.

Wolffe and Clump (1963) measured the turbulent velocity profiles in an eccentric annulus of $r_{e_i} = 1$ in. (25.4 mm) and $r_{e_o} = 1.5$ in. (38.1 mm) with an eccentricity ratio of $\epsilon = 0.25$. Using Eqs. 11 and 12, η_i and η_o are calculated as 2.280 and 1.887, respectively. Taking $a + d = 10$ in. (254 mm), the inner and outer radii of the corresponding concentric annulus system are determined using Eq. 13 as 1.023 and 1.515 in. (25.984 and 38.481 mm), respectively. Substitution of these values into Eq. 22 gives $r_{c_{max}} = 1.261$ in. (32.029 mm). Using Eq. 13, the corresponding constant η line that will give the maximum velocity locus in an eccentric annulus is determined to be 2.07. As can be seen from Figure 3, the experimental data of Wolffe and Clump lie on the theoretical locus of maximum velocity. Note that for $\epsilon = 0.25$ and $r_e^* = 2/3$, Eq. 19 also gives $\eta_{max} = 2.07$.

Velocity Distribution

In the preceding section the determination of the maximum velocity locus in an eccentric annulus is explained in detail. However, it should be noted that contrary to the case of a concentric system, the magnitude of the maximum velocity, v_{max}, varies along this locus. Therefore, the inversion technique alone cannot be used to determine the velocity distribution. The values of v_{max} on the maximum velocity locus should be determined experimentally as well.

Experimental velocity profiles can be obtained by taking measurements throughout the whole flow field. On the other hand, as explained later, the application of the inversion technique requires the measurements to be taken only on the maximum velocity locus. This leads to a significant reduction in the experimental task.

It is a common practice to express velocity distributions in eccentric annular ducts in the form of contour diagrams (Jonsson and Sparrow, 1966; Usui and Tsuruta, 1980). The contour diagram for a specified value of v_z/v^*, where v^* is the maximum velocity at the axis of symmetry (i.e., at $\xi = 0$), can be generated as follows:

1. For a given eccentric annulus system, determine the corresponding concentric system.
2. Calculate $r_{c_{max}}$ and the corresponding η_{max}.
3. Measure the values of v_{max} on η_{max} experimentally. (In the case of a laminar flow, v_{max} values can be calculated directly from Eq. 14.)
4. For a certain value of v_{max}/v^* on the locus of maximum

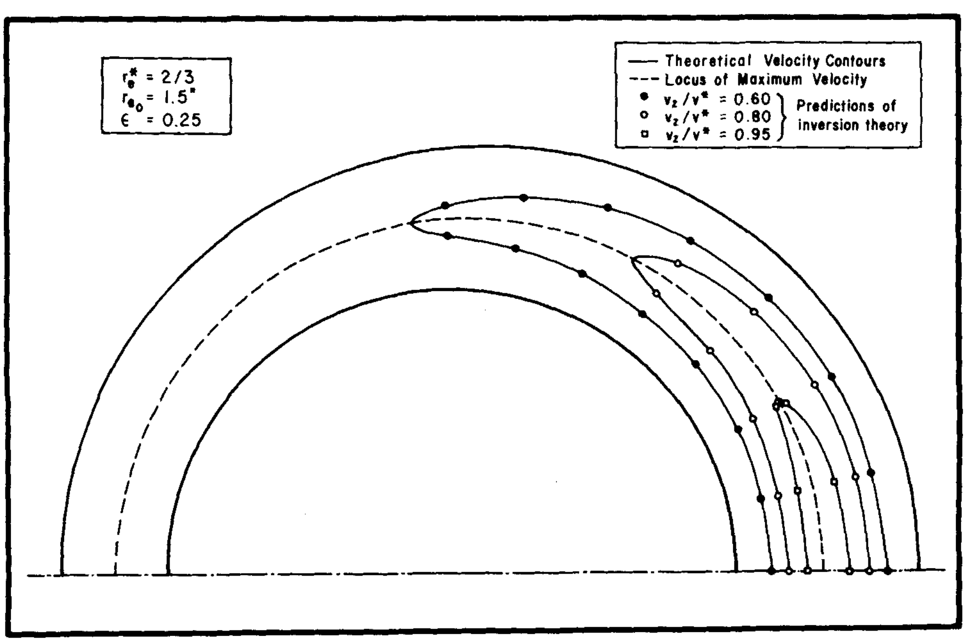

Figure 4. Velocity contour diagram for laminar flow.

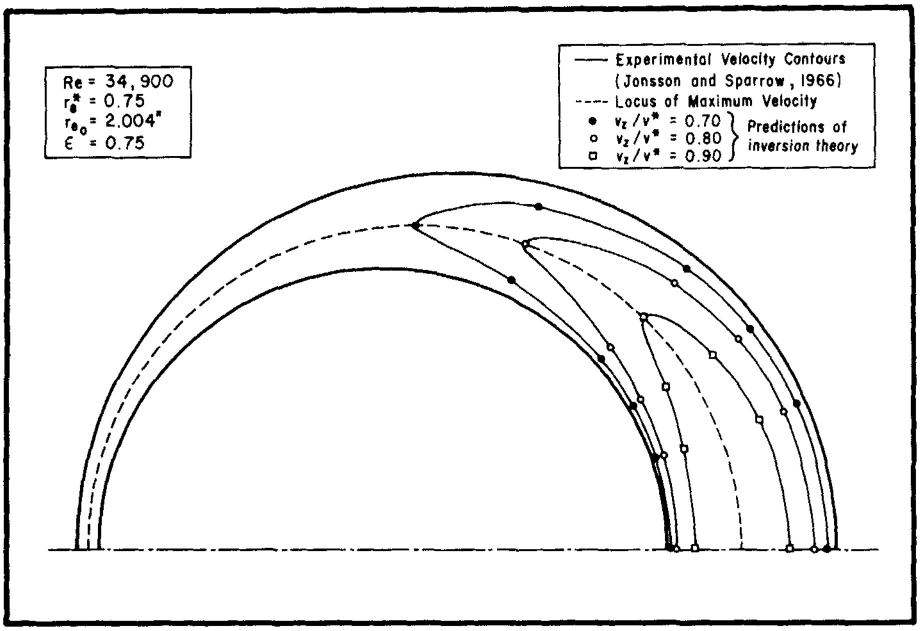

Figure 5. Velocity contour diagram for turbulent flow.

velocity, noting that $\eta = \eta_{max}$ on the locus, determine the ξ line passing through this point using Eq. 1.

5. To find the turbulent velocity profile in a concentric annulus, analytical expressions are needed. For this purpose the equations proposed by Knudsen and Katz (1958) are rearranged in the form

$$r = r_{c_o} - (r_{c_o} - r_{c_{max}}) \left(\frac{v_z/v^*}{v_{max}/v^*} \right)^{7.042} \quad \text{for} \quad r > r_{c_{max}} \qquad (24)$$

$$r = r_{c_i} + (r_{c_{max}} - r_{c_i}) \left(\frac{v_z/v^*}{v_{max}/v^*} \right)^{9.804} \quad \text{for} \quad r < r_{c_{max}} \qquad (25)$$

In the case of a laminar flow, r values can be calculated from the following equation using Newton-Raphson iteration:

$$(r/r_{c_o})^2 - 2\gamma \ln (r/r_{c_o}) = 1 - (1 - \gamma + \gamma \ln \gamma) \left(\frac{v_z/v^*}{v_{max}/v^*} \right) \qquad (26)$$

where

$$\gamma = \frac{1 - r_c^{*2}}{2 \ln (1/r_c^*)} \qquad (27)$$

After calculating the values of r, determine the corresponding η values using Eq. 13.

6. Repeat steps 4 and 5 for another value of v_{max}/v^* on the locus of maximum velocity.

The velocity contour diagrams for laminar and turbulent flows are shown in Figures 4 and 5, respectively. The results are in perfect agreement with theoretical and experimental data.

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Notation

a = dimensionless x coordinate of pole at $\eta = \infty$
B, C = constants, Eqs. 15 and 16
dP/dz = pressure gradient
e = eccentricity
F_n = constant, Eq. 17
G_n = constant, Eq. 18
r = radius
r^* = radius ratio
v_{max} = maximum velocity
v_z = axial velocity
v^* = maximum velocity at axis of symmetry ($\xi = 0$)
x, y, z = rectangular coordinates

Greek letters

γ = function, Eq. 27
ϵ = eccentricity ratio, Eq. 10
η, ξ = bipolar coordinates
μ = viscosity

Subscripts

c = concentric annulus
e = eccentric annulus
i = inner pipe
max = maximum
o = outer pipe

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